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Uniqueness of the Newton-Wigner Position Operator

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It is shown that the quantum position operator of Newton and Wigner for nonzero-mass systems is uniquely determined if one imposes a quantum "manifest covariance" condition of the same type as the similar condition of Currie, Jordan, and Sudarshan in the framework of the Hamiltonian formalism.

1. INTRODUCTION

The notion of localizability of relativistic particles has attracted considerable attention. It is generally admitted that the proper mathematical setting is the one due to Wightman (1962) (see also Varadarajan, 1985) which originates from the physical ideas of Newton and Wigner (1949).

Let us outline briefly this framework. We admit that the configuration space of a certain physical system is the Borel space (Q, β); here β is the Borel structure on Q. We also consider that the system is a pure quantum one. Then the lattice of the system is of the form $P(H)$; here H is some Hilbert space and $P(H)$ is the lattice of orthogonal projectors in H (see, for instance, Varadarajan, 1985).

Then one can argue (Wightman, 1962, p. 847) that the position observable is a projection-valued measure:

$$\beta \ni E \to P_E \in P(H)$$

The physical interpretation is the following: the states in the range of P_E correspond to the localization of our system in $E \subseteq Q$.

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If G is a (Borel) group of symmetries, then there exists in H a projective unitary representation of G:

$$G \ni \rightarrow U_g \in U(H)$$

Here $U(H)$ is the set of unitary operators in H.

If Q is a G-space, then a natural compatibility condition is

$$U_g P_E U_g^{-1} = P_{g \cdot E} \tag{1.1}$$

Here $g \cdot E$ is the image of E under the action of G. The couple (U, P) is called a system of imprimitivity for G based on Q.

If P is a position observable on $Q = R^3$, then according to the spectral theorem, we can construct three self-adjoint commuting operators Q_1, Q_2, and Q_3, namely the position operators appearing in the Newton–Wigner analysis. A convenient way to construct them (Varadarajan, 1985) is to define first the unitary operators

$$B(\mathbf{y}) \equiv \int \exp(-i\mathbf{x} \cdot \mathbf{y}) dP(\mathbf{x}) \tag{1.2}$$

for any $\mathbf{y} \in R^3$. Then the Q_k $(k = 1, 2, 3)$ are determined via Stone's theorem by

$$Q_k f \equiv i \frac{d}{dt} B(t\mathbf{e}_k) f|_{t=0} \tag{1.3}$$

Here $\mathbf{e}_k (k = 1, 2, 3)$ is the canonical basis in R^3.

A very interesting situation appears when one tries to combine these ideas with relativistic invariance. Suppose that our system admits a group of invariance G which has as a subgroup the special Euclidean group $SE(3)$. Then, we must have a projective unitary representation of G in H. By restrictions, we have a projective unitary representation U of $SE(3)$ in H. One says that the system is localizable (in R^3) iff there exists a position observable P (based on R^3) such that (U, P) is a system of imprimitivity.

According to the Newton–Wigner–Wightman analysis, the nonzero mass systems are localizable and zero-mass systems are not localizable in this sense. The nonzero- and zero-mass systems are by definition certain irreducible unitary continuous representations of the Poincaré (or the Galilei) group. The nonlocalizability for zero-mass systems, especially for the photon, has generated a rather extensive literature (Kalnay, 1971; Jauch and Piron, 1967; Amrein, 1969; Angelopolous *et al.*, 1974; Krauss, 1977; Jadzyk and Jancewicz, 1973; Bacry, 1988).

On the other hand, even in the case of nonzero-mass systems (which are localizable) a curious phenomenon appears, namely the position observable is not unique. In some cases this arbitrariness can be explicitly described

(Varadarajan, 1985). Namely, suppose that P^0 is a position observable of finite multiplicity and P is an arbitrary position observable. Then, there exists a unitary operator A in H such that A commutes with the representation U of $SE(3)$ and

$$P = AP^0A^{-1} \tag{1.4}$$

This nonunicity problem seems to be considered not very bothersome. In Varadarajan (1985) one can find a side remark to the sense that this problem is "noteworthy" and there exists a particular choice of P which seems to be the "simplest" and which gives the classical relationship between velocity and momentum.

On the other hand, it is known that a similar unicity problem can be solved in the framework of classical Hamiltonian mechanics, imposing the so-called "manifest covariance" conditions (Currie *et al.*, 1963). Namely, if K_i ($i = 1, 2, 3$) are the (Lorentz) boost generators and H is the Hamiltonian, then one can show that Q_1, Q_2, and Q_3 behave as the spatial components of a quadrivector iff one has the relation

$$\{K_i, Q_j\} = Q_i\{H, Q_j\} \tag{1.5}$$

for $i = 1, 2, 3$. It is tempting to use something similar in quantum mechanics. Jordan and Mukunda (1963) apply the naive correspondence rules:

$$\{\cdot, \cdot\} \to i[\cdot, \cdot] \tag{1.6}$$

$$AB \to \tfrac{1}{2}(\hat{A}\hat{B} + \hat{B}\hat{A}) \tag{1.7}$$

and obtain for the quantum operators K_i, H and Q_i the relation

$$[K_i, Q_j] = \tfrac{1}{2}(Q_i[H, Q_j] + [H, Q_j]Q_i) \tag{1.8}$$

Surprisingly, this equation gives a unique solution for systems of zero spin and has no solution for systems of nonzero spin.

In this note, we propose a different solution. Namely, we will impose a "manifest covariance" condition by a purely quantum argument, without invoking the correspondence rules (1.6) and (1.7). We will find a unique position operator for any spin, which is exactly the one considered in Varadarajan (1985) as the "simplest" one. Moreover, we will show that the relation (1.8) is valid up to terms of order $\hbar c^{-2}$. This quantum correction is zero iff the spin is zero. This completely explains the result of Jordan and Mukunda (1963). Moreover, this indicates that for the Galilei group, this quantum correction must disappear.

There have been a number of attempts to define a Lorentz covariant quadrivector position operator (see, e.g., Bertrand, 1973), but they differ from ours in motivation and outcome.

In Sections 2 and 3 we analyze the cases of Poincaré and Galilei invariance, respectively, and in Section 4 we formulate some conclusions.

2. THE POSITION OPERATOR FOR NONZERO-MASS POINCARÉ SYSTEMS

2.1. In the notations of Varadarajan (1985), the system $[m, s]$ of mass m and spin s corresponds to the irreducible projective representation $W^{m,+,s}$ of the proper orthochronous Poincaré group $P_+^\uparrow$.

We can realize this representation in the Hilbert space

$$H = L^2(R^3, C^{2j+1}, d\mathbf{p}/E(\mathbf{p}))$$

where

$$E(\mathbf{p}) \equiv (\mathbf{p}^2 + m^2)^{1/2} \tag{2.1}$$

as follows. Let X_m^+ be the positive-energy hyperboloid:

$$X_m^+ \equiv \{p \in R^4 \mid p_0^2 - \mathbf{p}^2 = m^2, p_0 > 0\} \tag{2.2}$$

We define $\tau: R^3 \to X_m^+$ by

$$\tau(\mathbf{p}) \equiv (E(\mathbf{p}), \mathbf{p}) \tag{2.3}$$

and $\eta: X_m^+ \to R^3$ by

$$\eta(p) \equiv \mathbf{p} \tag{2.4}$$

Then we have for any $a \in R^4$ and any $A \in SL(2, C)$

$$(W_{A,a}^{m,+,s} f)(\mathbf{p}) = e^{ia \cdot \eta(\mathbf{p})} D^{(s)}(U(A, \mathbf{p})) f(\eta(\delta(L)^{-1}(\tau(\mathbf{p})))) \tag{2.5}$$

for any $f \in H$. Here $U(A, \mathbf{p}) \in SU(2)$ is the so-called Wigner rotation, $D^{(s)}$ is the representation of weight s of $SU(2)$, and $\delta: SL(2, C) \to L_+^\uparrow$ is the covering homomorphism.

2.2. As we have said in the Introduction, the system $[m, s]$ is localizable in the sense of Newton and Wigner. Moreover, one can describe the most general expression of the position observable. More conveniently, one can describe the most general expression for the operators $B(\mathbf{y})$ defined by (1.2). Namely, one finds that (Varadarajan, 1985)

$$(B(\mathbf{y}) f)(\mathbf{p}) = A(\mathbf{p}) A(\mathbf{p} + \mathbf{y})^{-1} \left(\frac{E(\mathbf{p})}{E(\mathbf{p} + \mathbf{y})} \right)^{1/2} f(\mathbf{p} + \mathbf{y}) \tag{2.6}$$

Here $A: R^3 \to C^{2s+1}$ is a Borel function satisfying for any $U \in SU(2)$ almost everywhere

$$D^{(s)}(U)A(\mathbf{p}) \equiv A(\delta(U)\mathbf{p})D^{(s)}(U) \tag{2.7}$$

and $\delta: SU(2) \to SO(3)$ is the canonical homomorphism. Then, according to (1.3), one finds the following expression for the position operators Q_k:

$$(Q_k f)(\mathbf{p}) = i\frac{\partial f}{\partial p_k}(\mathbf{p}) - i\frac{p_k}{E(\mathbf{p})}\left[\frac{1}{2E(\mathbf{p})^2} + \frac{\partial A}{\partial p_k}(\mathbf{p})A(\mathbf{p})^{-1}\right]f(\mathbf{p}) \tag{2.8}$$

2.3. We formulate now the condition of relativistic covariance for the position operator. First, we start with the kinematic classical picture. Suppose we have two observers O and O' and they are connected by a boost of velocity $v = \mathrm{th}(\chi)$ in the direction $\mathbf{e}_3$. If O' sees the system at $t' = 0$ at the position $\mathbf{x}' = (x_1', x_2', x_3')$, then, according to the Lorentz rules of transformation, O sees the system localized at $\mathbf{x} = (x_1, x_2, x_3)$ at time t, where

$$t = \mathrm{th}(\chi)x_3, \qquad x_1' = x_1, \qquad x_2' = x_2, \qquad x_3' = \frac{1}{\mathrm{ch}(\chi)}x_3 \tag{2.9}$$

When we try to formulate this result in the quantum framework we are faced with the well-known problem that there are no states in H strictly localized in $\mathbf{x}$. More precisely, the eigenvalue equation

$$Q_k \psi_{\mathbf{x}} = x_k \psi_x \qquad (k = 1, 2, 3) \tag{2.10}$$

has solution of the form

$$\psi_{\mathbf{x}}(\mathbf{p}) = E(\mathbf{p})A(\mathbf{p})v\, e^{-i\mathbf{x}\cdot\mathbf{p}} \tag{2.11}$$

which is not an element of H. Here v is an arbitrary vector from C^{2s+1}.

Nevertheless, according to the quantum point of view, we can consider states which are localized at $\mathbf{x}$ in an approximate sense. For instance, let us define the states $\psi_{\mathbf{x},\alpha,v} \in H$ for any $\alpha \in R_+$ and any $v \in C^{2s+1}$ by

$$\psi_{\mathbf{x},\alpha,v}(\mathbf{p}) = \sqrt{2}\left(\frac{2\alpha}{\pi}\right)^{3/4} E(\mathbf{p})^{1/2}\, e^{-\alpha\mathbf{p}^2} A(\mathbf{p})v\, e^{-i\mathbf{x}.\mathbf{p}} \tag{2.12}$$

for $v \in C^{2s+1}$ of norm 1. Then it is easy to prove that

$$\|\psi_{\mathbf{x},\alpha,v}\|^2 = 1 \tag{2.13}$$

and

$$\langle q_k \rangle_{\psi_{\mathbf{x},\alpha,v}} = x_k \tag{2.14}$$

$$\Delta q_k = \alpha^{1/2} \tag{2.15}$$

So, in the limit $\alpha \to 0$, $\psi_{\mathbf{x},\alpha,v}$ becomes a state more and more localized at $\mathbf{x} \in R^3$.

2.4. Now, let $\psi_{\mathbf{x}',\alpha,v}$ be a state "approximately" localized at $\mathbf{x}'$. We consider that this measurement is done by the observer O' at time $t'=0$. (See Section 2.3 for notations.) According to the physical interpretation of the representation $W^{m,+,s}$, from the point of view of the observer O, the state of the system at $t=0$ is

$$W^{m,+,s}_{A_B(-\chi,\mathbf{e}_3),0}\psi_{\mathbf{x}',\alpha,v}$$

where $A_B(\chi, \mathbf{n}) \in SL(2, C)$ corresponds to the boost in the direction $\mathbf{n}$ of velocity $v = \text{th}(\chi)$. The discussion in Section 2.3 leads us to require that the observer O sees the particle localized around $\mathbf{x}$ at time $t = \text{th}(\chi)x_3$. But the state of the system at time $t = \text{th}(\chi)x_3$ is, from the point of view of O,

$$\psi_{\mathbf{x},\alpha,v,\chi} \equiv W^{m,+,s}_{1,\text{th}(\chi)x_3 e_0} W^{m,+,s}_{A_B(-\chi,\mathbf{e}_3),0}\psi_{\mathbf{x}',\alpha,v} \tag{2.16}$$

So, our condition of manifest covariance is that $\psi_{\mathbf{x},\alpha,v,\chi}$ describes a state localized around $\mathbf{x}$, where the connection between $\mathbf{x}$ and $\mathbf{x}'$ is given by (2.9). Namely, we must have

$$\langle q_k \rangle_{\psi_{\mathbf{x},\alpha,v,\chi}} = x_k \tag{2.17}$$

and

$$\lim_{\alpha \to 0} \Delta q_k = 0 \tag{2.18}$$

for $k = 1, 2, 3$ and any $v \in C^{2s+1}$.

Let us note that if (2.17) and (2.18) are true, then because of the rotation covariance, a similar statement will be true for boosts in any direction.

Now, it is not hard to prove that (2.17) is true for $k = 1, 2$ and is also true for $k = 3$ iff $A = \text{const}$ almost everywhere. So we can take $A = 1$. In this case it follows easily that Δq_k behaves as $\alpha^{1/2}$ for small α, so (2.18) follows.

In conclusion, our main result is that the manifest covariance conditions formulated above is compatible with a *unique* expression for the position operator, namely

$$(Q_k f)(\mathbf{p}) = i\frac{\partial f}{\partial p_k}(\mathbf{p}) - i\frac{p_k}{2E(\mathbf{p})^2} f(\mathbf{p}) \tag{2.19}$$

2.5. To study the validity of (1.4) it is easier to work in the vector bundle representation for $W^{m,+,s}$ (see Varadarajan, 1985, p. 365).

Then the operators Q_k are given by the formula (VIII.230) of Varadarajan (1985). One can easily compute the infinitesimal generators K_l ($l = 1, 2, 3$)

in this representation. Then, inserting appropriately the factors $\hbar$ and c, one can show that we have

$$[K_i, Q_j] = \tfrac{1}{2}(Q_i[H, Q_j] + [H, Q_j]Q_i) + o(\hbar c^{-2}) \tag{2.20}$$

where $o(\hbar c^{-2})$ is an expression containing only spin terms. This is the quantum version of the "manifest covariance" condition of Currie *et al.* (1963).

3. THE POSITION OPERATOR FOR NONZERO-MASS GALILEI SYSTEM

3.1. In the notations of Varadarajan (1985), the system of mass m and spin s corresponds to the representation $V^{\tau,s}$ of the covering group of the proper orthochronous Galilei group $G_+^\uparrow$. It can be realized in the Hilbert space $H = L^2(R^3, C^{2s+1}, d\mathbf{p})$, according to the following formula:

$$(V^{\tau,s}_{U,\mathbf{u},\eta,\mathbf{a}} f)(\mathbf{p}) = \exp\left(i\mathbf{a}\cdot\mathbf{p} + \frac{i}{2}\tau\mathbf{a}\cdot\mathbf{u} - i\eta\gamma\frac{\mathbf{p}^2}{2\tau}\right) D^{(s)}(U) f(\delta(U)^{-1}(p + \tau\mathbf{u})) \tag{3.1}$$

Here $U \in SU(2)$, $\mathbf{u} \in R^3$ is the velocity, $\mathbf{a}$ is the space translation, and $\eta \in R$ is the time translation.

As in Section 2, we can describe explicitly the most general position observable. Namely, the operators $B(\mathbf{x})$ must have the following generic form:

$$(B(\mathbf{x}))f(\mathbf{x}) = A(\mathbf{p})A(\mathbf{p}+\mathbf{x})^{-1}f(\mathbf{p}+\mathbf{x}) \tag{3.2}$$

where $A: R^3 \to C^{2s+1}$ is a Borel function satisfying (2.7). It follows that the position operators are

$$(Q_k f)(\mathbf{p}) = i\frac{\partial f}{\partial p_k}(\mathbf{p}) - i\frac{\partial A}{\partial p_k}(\mathbf{p})A(\mathbf{p})^{-1}f(\mathbf{p}) \tag{3.3}$$

3.2. The condition of manifest covariance with respect to Galilei boosts is simpler than in the Poincaré case. Let us suppose that the two observers O and O' are connected by a boost of velocity $\mathbf{u}$. If O sees the system at $t = 0$ at the position $\mathbf{x}$, then according to the formula for the Galilei boosts, the observer O' sees the system at $t' = 0$ at the position $\mathbf{x}' = \mathbf{x} - t\mathbf{u} = \mathbf{x}$.

As in Section 2, the solution of the eigenvalue equation

$$Q_k \psi_{\mathbf{x}} = x_k \psi_{\mathbf{x}} \tag{3.4}$$

($k=1, 2, 3$) is of the form

$$\psi_{\mathbf{x},v}(\mathbf{p})=e^{-i\mathbf{x}\cdot\mathbf{p}}A(\mathbf{p})v \tag{3.5}$$

for any $v\in C^{2s+1}$, and so is not an element of the Hilbert space H. Nevertheless, we can consider the elements

$$\psi_{\mathbf{x},\alpha,v}(\mathbf{p})=\left(\frac{\pi}{2\alpha}\right)^{3/2} e^{-i\mathbf{p}\cdot\mathbf{x}}\, e^{-\alpha\mathbf{p}^2}A(\mathbf{p})v \tag{3.6}$$

for $v\in C^{2s+1}$ of norm 1 and we have

$$\langle q_k\rangle_{\psi_{\mathbf{x},\alpha,v}}=x_k \tag{3.7}$$

and

$$\Delta q_k=\alpha^{1/2} \tag{3.8}$$

for $k=1, 2, 3$.

3.3. Now, let $\psi_{\mathbf{x},\alpha,v}$ be the state of the system from the point of view of the observer O'. As in Section 2, the state of the system from the point of view of the observer O is

$$\psi_{\mathbf{x},\alpha,\mathbf{u},v}\equiv V^{\tau,s}_{1,-\mathbf{u},0,0}\psi_{\mathbf{x},\alpha,v} \tag{3.9}$$

According to the discussion in Section 3.2, our condition of manifest covariance is that $\psi_{\mathbf{x},\alpha,\mathbf{u},v}$ describes a system localized around $\mathbf{x}$ at $t=0$, i.e.,

$$\langle q_k\rangle_{\psi_{\mathbf{x},\alpha,\mathbf{u},v}}=x_k \tag{3.10}$$

and

$$\lim_{\alpha\to 0}\Delta q_k=0 \tag{3.11}$$

for $k=1, 2, 3$.

It is elementary to prove that (3.10) is equivalent to $A=\text{const}$, so, as in Section 2, we can take $A=1$. So, in this case also, the manifest covariance condition gives a unique position operator, namely

$$(Q_k f)(\mathbf{p})=i\frac{\partial f}{\partial p_k}(\mathbf{p}) \tag{3.12}$$

It is easy to see that in this case we have

$$[K_i, Q_j]=0 \tag{3.13}$$

Again we note that (3.13) is the quantum counterpart of a similar classical relation (Jordan and Mukunda, 1963). In this case there are no quantum corrections. This can be explained by the presence of c^{-2} in the expression of the relativistic quantum correction in (2.20).

4. CONCLUSIONS

The relation (2.20) shows that the naive quantization rules (1.6) and (1.7) do not always work. This explains in part the difficulties of all the quantization schemes used in the literature.

It is plausible that the same unicity result holds also for the photon if the localizability of this system is described in the sense of the Wightman framework (Grigore, 1989).

It is interesting to try to generalize along these lines the well-known results of the 'noninteraction"-type theorems (Jordan and Mukunda, 1963; Bertrand, 1973).

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